

# Irreversible energy flow in forced Vlasov dynamics

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**Abstract.** The recent paper of Plunk [2013] considered the forced linear Vlasov equation as a model for the quasi-steady state of a single stable plasma wavenumber interacting with a bath of turbulent fluctuations. This approach gives some insight into possible energy flows without solving for nonlinear dynamics. The central result of the present work is that the forced linear Vlasov equation exhibits asymptotically zero (irreversible) dissipation to all orders under a detuning of the forcing frequency and the characteristic frequency associated with particle streaming. We first prove this by direct calculation, tracking energy flow in terms of certain exact conservation laws of the linear (collisionless) Vlasov equation. Then we analyze the steady-state solutions in detail using a weakly collisional Hermite-moment formulation, and compare with numerical solution. This leads to a detailed description of the Hermite energy spectrum, and a proof of no dissipation at all orders, complementing the collisionless Vlasov result.

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## 1 Introduction

In this paper we revisit the equation considered by Plunk [2013], *i.e.* the forced one-dimensional Vlasov equation for an electrostatic quasi-neutral plasma. In studying this equation, we hope to gain insight into the more difficult problem of how Landau damping behaves in a turbulent setting. The basic premise is as follows. If a single linearly stable plasma wavenumber is participating in a turbulent steady state, involving many other wavenumbers, then the nonlinear term in the equation for this component must behave as a statistically stationary source, *i.e.* its statistics do not depend on time. By the Wiener–Khinchin theorem, this means that the Fourier components of the signal can be considered uncorrelated, and the statistical properties of the component (and its interaction with the source) can be deduced from its exact response to a single frequency drive – so-called “harmonic forcing”.

This work originates from an interest in gyrokinetic turbulence, in which the free streaming of plasma particles along a magnetic guide field gives rise to linear Landau damping. However, this occurs simultaneously to the nonlinear cascade of “free energy” that interferes with the Landau damping. The gyrokinetic equation is applied to fusion experiments, but also to some astrophysical systems as well. However, the equations in this paper are simpler than the full gyrokinetic system, and so it is possible that the results have a more general applicability.

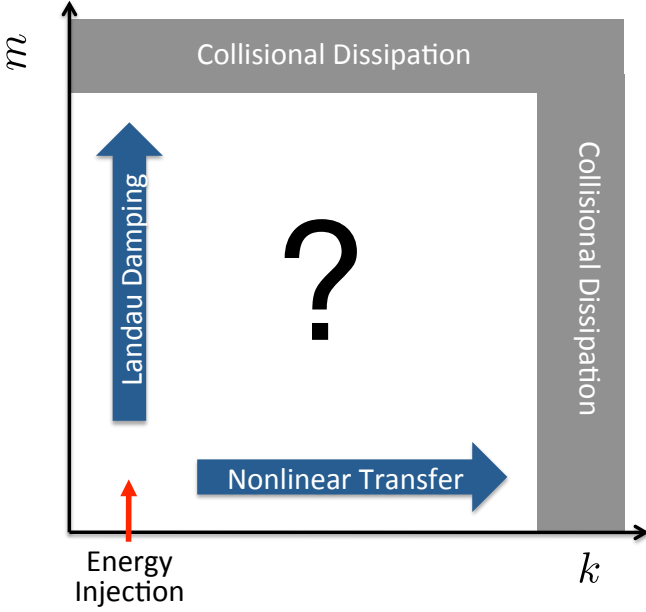
The chief subject of this paper is entropy production. We use this term in two senses. One is to emphasize that

Landau damping, a property of the (collisionless) Vlasov equation, represents an irreversible process. A second reason is to allude to the more narrowly defined notion of entropy production in a collisional plasma. That is, the Boltzmann equation (Vlasov equation plus collisions) satisfies Boltzmann’s H-theorem, whereby a specific (information theoretic) definition of entropy is a strictly increasing function of time. Physically, the “phase-mixing” process involved in Landau damping induces a flow of energy to small scales where in any finite representation (*i.e.* a numerical implementation) it must be “mopped up” by explicit dissipation like a collision operator. Thus, there is a clear correspondence between the flux of energy (in velocity-scale-space) and the collisional entropy production of Boltzmann’s H-theorem.

The problem of Landau damping in turbulence can be stated as a problem of dissipation. The basic question is: where does the energy go? Energy is injected by some source, is nonlinearly redistributed among wavenumbers, exciting waves (*i.e.* normal modes and Landau modes) and other degrees of freedom, and is ultimately routed to some dissipation channels. We may conceptually think of the energy flow in a two dimensional space, as in Fig. 1, where the horizontal axis represents the dimensions through which nonlinear flow of energy occurs, *e.g.* the wavenumber  $k$ , and the vertical axis represents an index for the modes excited by the Landau damping process, *e.g.* the Hermite index  $m$ .

There are two extreme ways in which the energy may flow in this plane. One was described by Howes et al. [2008] (but, it should be noted, merely for the purpose

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**Fig. 1.** Cartoon of turbulent energy flow via linear and non-linear channels.

of illustrating disagreement with observations). It can be summarized as follows. Since the flow of energy in this space is conservative, it can be represented by a simple continuity equation if it occurs locally (among neighboring wavenumbers). The direct integration of the equation, under the assumption that linear damping always competes with the nonlinear transfer rate, then leads to an exponentially decaying spectrum of energy in  $k$ . In the limit of weak collisionality, all of the energy flux is consumed by Landau damping, and no energy reaches large- $k$  dissipation mechanisms. However, this exponential decay is not observed and instead a power-law spectrum persists.

On the other hand, one could reason that no energy should flow into the Landau channel. This was argued in several papers including Schekochihin et al. [2008], Plunk et al. [2010], based on the fact that nonlinear energy transfer accelerates with  $k$  and so reaches high- $k$  dissipation more quickly than high- $m$  dissipation. The weakness in this argument is that damping is determined by energy flux, not the speed of flow in  $m$ -space. The fact that energy travels more slowly in the Landau channel can easily be attributed to shallowness of the spectrum. Indeed, when the spectrum of energy is non-integrable (e.g.  $W(m) \sim m^{-n}$ ,  $n \leq 1$ ) then energy can never dissipate in the limit of small collisionality; but *damping* would persist because the flux of energy through a finite- $m$  mode does not strongly depend on where or how the dissipation is achieved.

Clearly, simple arguments are not satisfactory to explain the role of Landau damping in kinetic plasma turbulence, and the issue remains fundamentally unresolved. The only thing that can be concluded *a priori* is that if the spectrum of turbulence is modified by energy loss to the Landau channel, then the resulting turbulent spec-

trum (i.e. the energy density in  $k$ -space) must be steeper than the unmodified spectrum.

In short, the discussion above motives us to look for simple mechanisms that can obstruct or enhance the Landau damping process. Linear Landau damping describes the decay of a single isolated wave that is left undisturbed for a sufficiently long time to approach a state of steady decay, whereby energy is drained from the wave via a route through successively smaller scales in velocity space. How is this process affected when disrupted by the presence of an external source? The remainder of the paper addresses this question.

## 2 Equations and basic analysis

We consider the one-dimensional electrostatic reduced Vlasov equation for a single Fourier component, under harmonic forcing

$$\frac{\partial f}{\partial t} + ikvf + ikc_s^2 n(t)G(v) = \exp(-i\Omega t)S(v), \quad (1)$$

where  $k > 0$  for simplicity. The density of  $f$  is

$$n(t) = \int_{-\infty}^{\infty} dv f(v, t). \quad (2)$$

This system can be derived from the full kinetic system (e.g. gyrokinetics) by integrating the kinetic equation over velocity coordinates that are not associated with the Landau damping. Note that for simplicity, we are neglecting magnetic fluctuations. However, in Appx. A we demonstrate how to generalize the analysis to include a fluctuating magnetic field.

We now apply the Morrison transform [Morrison and Pfirsch, 1992, Morrison, 2000] to Eqn. 1 using the notation of Plunk [2013]:

$$\frac{\partial \tilde{f}}{\partial t} + iku\tilde{f} = \exp(-i\Omega t)\tilde{S}(u), \quad (3)$$

where the transformed distribution is defined

$$\tilde{f}(u) = \frac{f_+(u)}{D_+(u)} + \frac{f_-(u)}{D_-(u)}, \quad (4)$$

and we define  $D_{\pm}(u) = 1 \pm 2\pi i c_s^2 G_{\pm}(u)$ . The positive- and negative-frequency parts of an arbitrary function  $h(v)$  are defined in terms of the Fourier transform by

$$h_{\pm}(u) = \pm \int_0^{\pm\infty} dv e^{ivu} \int_{-\infty}^{\infty} dv \frac{e^{-ivv}}{2\pi} h(v). \quad (5)$$

Note that  $h(u) = h_+(u) + h_-(u)$  and  $D_+ = D_-^*$  with superscript  $*$  denoting the complex conjugate. We will also need the Hilbert transform, which for an arbitrary function  $h$  is denoted with a subscript  $*$ :

$$h_*(u) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{h(v)dv}{u-v}. \quad (6)$$

where  $P$  denotes the principal value. Following Plunk [2013] the long-time ( $t \rightarrow \infty$ ) solution of Eqn. 3 can be written as

$$\tilde{f}(u, t) = \exp(-i\Omega t) \tilde{S}(u) K(u - \Omega/k)/k, \quad (7)$$

where

$$K(x) = \pi \left[ \delta(x) - (i/\pi) P \frac{1}{x} \right], \quad (8)$$

which, using identity 59, implies that  $\int_{-\infty}^{\infty} dx K(x-y) h(x) = \pi(h + ih_*) = 2\pi h_+(y)$  for arbitrary  $h$ . Then we may compute  $f(v)$  by the inverse of the Morrison transform:

$$f(v, t) = \int_{-\infty}^{\infty} du \tilde{f}(u, t) f^u(v), \quad (9)$$

where  $f^u(v) = \lambda_u \delta(u - v) + c_s^2 P[G(v)/(u - v)]$ , where  $P$  denotes the principal value with respect to the point  $u = v$ , and  $\lambda_u = 1 - c_s^2 \pi G_*(u)$ . We thus find

$$f(v, t) = \exp(-i\Omega t) \{ (1 - \pi c_s^2 G_*) \tilde{f}_\infty - \pi c_s^2 G \tilde{f}_{\infty*} \}, \quad (10)$$

where  $\tilde{f}_\infty = \tilde{S}(v) K(v - \Omega/k)/k$ .

### 3 “No Entropy Production” (NEP) Theorem

In Plunk [2013], damping was quantified by the wave-particle interaction term (*i.e.* the electric field multiplied by the plasma current) directly. Although this is an intuitive measure, and allows a direct comparison with Landau damping, it is illuminating to consider energy flow in a more general sense, via the conservative properties of the Vlasov system. In the absence of forcing it is easy to see that

$$\mathcal{W}(u) = |\tilde{f}|^2/2, \quad (11)$$

is conserved, as is any quantity formed from  $\mathcal{W}$  by arbitrarily weighted integration over  $u$ . In the presence of a source, the balance equation is

$$\begin{aligned} \frac{d\mathcal{W}}{dt} &= \text{Re}[\tilde{f}^* \tilde{S}] \\ &= \frac{1}{k} |\tilde{S}|^2 \text{Re}[K(u - \Omega/k)] \\ &= \frac{\pi}{k} |\tilde{S}|^2 \delta(u - \Omega/k), \end{aligned} \quad (12)$$

where we have used the steady state solution for  $\tilde{f}$ , Eqn. 7, between the first line and second line, and  $\text{Re}[K(x)] = \delta(x)$  between the second line and third line. Unsurprisingly, this quasi-steady state is singular, but by (arbitrarily) integrating over  $u$  we can extract well-defined measures of input “power”. Such integrals generally fall off sharply at small  $kv_T/\Omega$  as expected from the suppression of damping found in Plunk [2013]. This can be seen by rewriting

$$\begin{aligned} \tilde{S} &= \frac{1}{|D|^2} (D^* S_+ + D S_-) \\ &= \frac{1}{|D|^2} (S \text{Re}[D] + S_* \text{Im}[D]), \end{aligned} \quad (13)$$

where we have used  $D_- = D_+^*$  and written  $D_+ = D$ , and used the identity 59 between the first and second line. Identity 59 also implies

$$\text{Re}[D] = 1 - \pi c_s^2 G_*, \quad (14)$$

$$\text{Im}[D] = \pi c_s^2 G. \quad (15)$$

Note that an important property of  $G_*$  is that it decays algebraically if  $G(v)$  decays super-algebraically. This is because the integral in Eqn. 6, which is dominated by  $v \sim v_T$ , may be expanded in powers of  $kv_T/\Omega$  (in analogy to a multi-pole expansion to calculate the electric field far from a compact charge distribution). Thus, Eqns. 13-15 imply that if  $S(u)$  and  $G(u)$  fall off faster than any power of  $u$ , then  $\tilde{S}(\Omega/k)$  falls off faster than any power of  $\Omega/k$ , as does the input power of any energy quantities formed by integrating  $\mathcal{W}$  over  $u$ . However, the quantity  $\mathcal{W}$  is rather abstract – what is its physical meaning? It is also not clear precisely what Eqn. 12 implies for collisional dissipation. Let us turn to a more conventional definition of energy. Assuming  $G$  has no zeros, Eqn. 1 with  $S = 0$  conserves the quantity

$$W = \int v dv \frac{|f|^2}{v_T^2 G} + \alpha |n|^2/2, \quad (16)$$

where  $\alpha = 2c_s^2/v_T^2$ . Now taking  $G = 2vf_M/v_T^2$ , this quantity is a reduced version<sup>1</sup> of the free energy, which, for a gyro- or drift-kinetic plasma with a homogeneous Maxwellian background, is an exact collisionless global invariant when summed over  $\mathbf{k}$ . The input of free energy  $W$  also determines entropy production in the weakly collisional limit. It is obvious that if the input of  $W$  tends to zero, so does that of  $W$ . However, the input rate of  $W$  can never be exactly zero, and so what is more important is how strongly it goes to zero. It turns out that the input rate of  $W$  goes to zero super-algebraically, for a general class of  $S$ . In particular assuming  $S(v) = S_0 \varphi(v) f_M(v)$  where  $\varphi = \sum \varphi_m v^m$ , we can calculate the rate of free energy injection to be

$$\frac{dW}{dt} = \frac{2\pi}{k} |S_0|^2 \text{Re} \left[ H_+ \left( \frac{\Omega}{k} \right) \right], \quad (17)$$

where

$$\text{Re}[H_+] = \text{Re} \left[ \frac{(\psi f_M)_+}{D_+} + \alpha \frac{f'_{M+}}{D_+} \Phi[\psi, f_M] \right], \quad (18)$$

and  $f'_M(v) = v f_M(v)$ . The details of this derivation are given in Appx. C. Because  $\psi(v)$ , given by Eqn. 76, is a

<sup>1</sup> Eqn. 1 is reduced to one-dimension in velocity space.

purely real polynomial, identity 59 implies that this expression goes to zero super-algebraically in  $kv_T/\Omega$ . To illustrate this, let us work out a simple case more explicitly, *i.e.* two-moment forcing ( $\varphi = 1 + cv$ ). Using  $f_{M+} = Z(\xi)/(2\pi i v_T)$ , where  $Z$  is the plasma dispersion function and  $\xi = \Omega/(kv_T)$ , we find

$$\frac{dW}{dt} = \frac{|S_0|^2}{kv_T} \operatorname{Re} \left[ \frac{Z + (A + B\xi)[1 + \xi(1 + \xi Z)]}{i[1 + \alpha(1 + \xi Z)]} \right], \quad (19)$$

where  $Z = Z(\xi)$ ,  $A = 2v_T \operatorname{Re}[c]$  and  $B = v_T^2 |c|^2 (1 - \alpha + \alpha^2/(1 + \alpha))$ . For large  $\xi$ , the Maxwellian case ( $c = 0$ ) yields  $dW/dt = |S_0|^2 (1 + \alpha)^2 \sqrt{\pi} \exp(-\xi^2)/(kv_T)$ , which can be compared with Fig. 2.

## 4 Formulation in Hermite Space

To facilitate numerical solution and to provide a different mathematical viewpoint, let us reformulate the problem as weakly collisional dynamics in Hermite space. We add a dissipation operator to the right hand side of Eqn. 1 and represent our distribution function with a Hermite series

$$f(v, t) = \sum_{m=0}^{\infty} a_m(t) \frac{H_m(\hat{v})}{\sqrt{2^m m!}} f_M, \quad (20)$$

where  $\hat{v} = v/v_T$  and the Hermite polynomials are

$$H_m(\hat{v}) = (-1)^m e^{\hat{v}^2} \frac{d^m}{d\hat{v}^m} (e^{-\hat{v}^2}). \quad (21)$$

This is a popular approach to solving the Vlasov equation, which gives an explicit definition of scale via the Hermite index  $m$ . The kinetic equation is now

$$\begin{aligned} \frac{\partial a_m}{\partial t} + ikv_T \left( \sqrt{\frac{m+1}{2}} a_{m+1} + \sqrt{\frac{m}{2}} a_{m-1} \right) \\ + ikc_s^2 G_m a_0 = S_m e^{-i\Omega t} - \nu m^n a_m \end{aligned} \quad (22)$$

where we have expanded  $S$  and  $G$  in Hermite moments (where  $G_m = \sqrt{2} \delta_{m1}/v_T$ ), and included an  $n$ -iterated Lénard–Bernstein collision operator. Linear phase mixing now takes the form of nearest-neighbor coupling in  $m$ -space, and we may track the flow of energy to “fine scales” (high- $m$ ) by directly examining these couplings and inspecting the steady-state  $m$ -spectrum.

### 4.1 Entropy production

The free energy balance can now be written

$$\frac{dW}{dt} = \dot{W}_S + \dot{W}_C, \quad (23)$$

$$W = \frac{\alpha}{2} |a_0|^2 + \frac{1}{2} \sum_{m=0}^{\infty} |a_m|^2 \quad (24)$$

$$\dot{W}_S = \operatorname{Re} \left( \alpha a_0^* S_0 e^{-i\Omega t} + \sum_{m=0}^{\infty} a_m^* S_m e^{-i\Omega t} \right), \quad (25)$$

$$\dot{W}_C = -\nu \sum_{m=0}^{\infty} m^n |a_m|^2. \quad (26)$$

Let us consider what happens to our system under frequency detuning  $\omega_{NL} > kv_T$ .

### 4.2 Numerical comparison and asymptotic solution

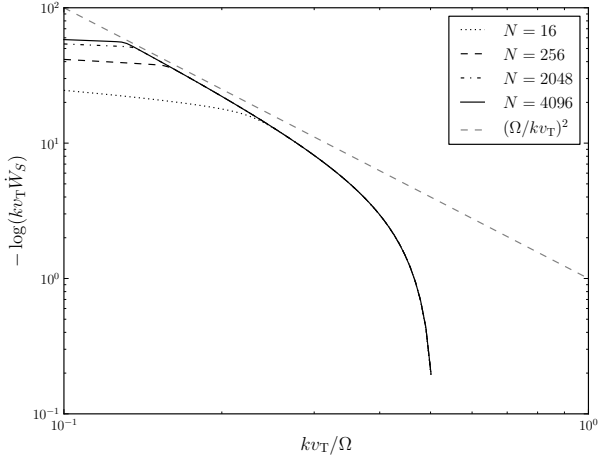
To verify Eqn. 17 and the NEP theorem, let us consider numerical solutions of Eqn. 22, truncating the Hermite expansion for  $f$  after  $N$  terms. In Fig. 2 we plot the long-time result  $-\log(|kv_T \dot{W}_S|)$  against  $kv_T/\Omega$  for the Maxwellian driving case  $S_m = \delta_{m0}$ . This has gradient  $-2$  in the limit  $kv_T/\Omega \rightarrow 0$ , indicating that  $\dot{W}_S \sim \exp(-(\Omega/kv_T)^2)/kv_T$ , in accordance with Eqn. 19. In general we may take any set of  $S_m$ , but the Maxwellian driving case is representative.

Each resolution  $N$  in Fig. 2 has a critical  $kv_T/\Omega$  below which the scaling does not hold. This is a numerical artefact, which is due to the sharp falloff in dissipation with decreasing  $kv_T/\Omega$ . To see this, note that the collisional numerical calculation should generally recover the collisionless result (Eqn. 17) in the limit  $\nu \rightarrow 0+$ , provided that resolution is simultaneously increased  $N \rightarrow \infty$  to retain high- $m$  dissipation. However, this is very costly for small  $kv_T/\Omega$ . This is simply because the input of free energy, and thus the dissipation in steady state, becomes extremely small and so dissipation at large scales becomes important unless  $\nu$  is also made very small. Very small  $\nu$ , of course, demands very large resolution  $N$ . The “corner” in Fig. 2 is the point where the low- $m$  and high- $m$  contributions to  $\dot{W}_C$  balance.

Now let us consider the Hermite spectra. These are shown in Fig. 3 for various  $k$  in the Maxwellian driving case. There are three distinct behaviors corresponding to low, medium and high  $m$ . The medium and high  $m$  case was described by Zocco and Schekochihin [2011]. For  $m \gg 1$  one may neglect driving and the Boltzmann response, and derive an energy equation from 22 where the streaming term is approximated with an  $m$ -derivative. Solving this gives the spectrum for large  $m$ ,

$$|a_m|^2 \propto \frac{1}{\sqrt{m}} \exp \left( -\frac{\nu}{kv_T} \frac{\sqrt{2} m^{n+1/2}}{2n+1} \right) \quad (27)$$

which is in excellent agreement with the spectra in Fig. 3. For medium  $m$  the spectrum behaves like  $1/\sqrt{m}$ , while for large  $m$  the exponential collision term dominates. Thus it



**Fig. 2.** Free energy injection against  $kv_T/\Omega$  for  $\alpha = 1$  and various resolutions  $N$ .

is unavoidable that the flow of energy ultimately assumes a conventional form of the autonomous (Landau) solution at sufficiently large  $m$ . However, the amount of dissipation depends on the overall amplitude of this Landau tail, which is determined by the small- $m$  behavior of the spectrum.

To gain insight into behavior at small  $m$ , we can solve the kinetic equation 22 neglecting collisions ( $\nu = 0$ ) in the long-time limit, *i.e.* taking  $a_m(t) = \hat{a}_m e^{-i\Omega t}$ . We assume the source only drives the first  $\bar{m}$  moments, such that  $S_m = 0$  for all  $m > \bar{m}$ . Eqn. 22 becomes

$$-i\Omega \hat{a}_m + ikv_T \mathbf{L} \hat{a}_m = S_m, \quad (28)$$

where  $\mathbf{L}$  is the operator

$$\mathbf{L} \hat{a}_m = \sqrt{\frac{m+1}{2}} \hat{a}_{m+1} + \sqrt{\frac{m}{2}} (1 + \alpha \delta_{m1}) \hat{a}_{m-1}. \quad (29)$$

Let us consider the limit  $kv_T \sqrt{m}/\Omega \ll 1$ , for which we can make analytical progress. We solve Eqn. 28 treating  $kv_T/\Omega$  as a small parameter. Setting

$$\hat{a}_m = \sum_{n=0}^{\infty} \hat{a}_m^{(n)} (kv_T/\Omega)^n, \quad (30)$$

we find that streaming vanishes at leading order and the equation is local in  $m$  with the solution

$$\hat{a}_m^{(0)} = \frac{iS_m}{\Omega}. \quad (31)$$

From Eqn. 28, higher orders are found iteratively with

$$\hat{a}_m^{(n)} = \mathbf{L} \hat{a}_m^{(n-1)}, \quad n \geq 1, \quad (32)$$

Thus we immediately find at zeroth order that  $\hat{a}_m = iS_m/\Omega$  for the forced components  $m \leq \bar{m}$ , while the unforced components,  $m > \bar{m}$ , are zero. Moreover as coupling is nearest-neighbor, every moment with  $m > \bar{m}$  is an order higher in  $kv_T/\Omega$  than the moment below it, *i.e.*

$$\begin{aligned} \hat{a}_{\bar{m}+1} &= \frac{kv_T}{\sqrt{2}\Omega} \sqrt{\bar{m}+1} (1 + \alpha \delta_{\bar{m}0}) \hat{a}_{\bar{m}}, \\ \hat{a}_{\bar{m}+2} &= \frac{k^2 v_T^2}{2\Omega^2} \sqrt{(\bar{m}+2)(\bar{m}+1)} (1 + \alpha \delta_{\bar{m}0}) \hat{a}_{\bar{m}}, \end{aligned} \quad (33)$$

and so forth, implying  $\hat{a}_m \sim \mathcal{O}(\hat{a}_{\bar{m}} (kv_T/\Omega)^{m-\bar{m}} \sqrt{m!/\bar{m}!})$ . Thus,  $\hat{a}_m$  falls off rapidly with  $m$  until  $kv_T \sqrt{m}/\Omega \sim \mathcal{O}(1)$  and streaming can no longer be neglected at leading order. The point of transition to the  $m^{-1/2}$  spectrum in Fig. 3 agrees well with  $kv_T \sqrt{m}/\Omega \sim \mathcal{O}(1)$ .

This solution also illustrates that free energy injection  $\dot{W}_S$  vanishes at every order in  $kv_T/\Omega$ . Let us first show this for  $\alpha = 0$ , and then generalize the solution. The  $n$ th-order contribution to  $\dot{W}_S$  is

$$\dot{W}_S^{(n)} = \text{Re} \left( \sum_{m=0}^{\infty} \hat{a}_m^{(n)*} S_m \right). \quad (34)$$

The  $n = 0$  case trivially vanishes, while for  $n \geq 1$

$$\dot{W}_S^{(n)} = \text{Re} \left( \frac{i}{\Omega} \sum_{m=0}^{\infty} (\mathbf{L}^n S_m^*) S_m \right). \quad (35)$$

By shifting the indices in the sum we may show that

$$\sum_{m=0}^{\infty} (\mathbf{L}^n S_m^*) S_m = \sum_{m=0}^{\infty} (\mathbf{L}^{n-1} S_m^*) \mathbf{L} S_m, \quad (36)$$

so that iterating

$$\dot{W}_S^{(n)} = \text{Re} \left( \frac{i}{\Omega} \sum_{m=0}^{\infty} S_m^* \mathbf{L}^n S_m \right), \quad (37)$$

and  $\dot{W}_S^{(n)}$  therefore vanishes as the sum is equal to its complex conjugate.

This proof relies on Eqn. 36 which follows from the symmetry that the two terms in  $S_m \mathbf{L} S_m^*$  are complex conjugates but with a shift in  $m$ . This symmetry is broken when  $\alpha \neq 0$ , but may be restored with the substitutions

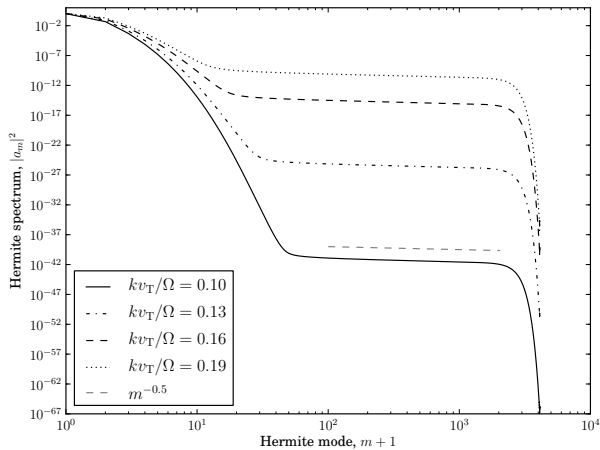
$$b_m = \begin{cases} \hat{a}_0 \sqrt{1+\alpha}, & m=0, \\ \hat{a}_m, & \text{else,} \end{cases} \quad (38)$$

$$S'_m = \begin{cases} S_0 \sqrt{1+\alpha}, & m=0, \\ S_m, & \text{else.} \end{cases} \quad (39)$$

The kinetic equation 28 becomes

$$-i\Omega b_m + ikv_T \mathbf{L}' b_m = S'_m, \quad (40)$$

where  $\mathbf{L}'$  is the symmetric operator



**Fig. 3.** Hermite spectra for  $N = 4096$  and  $\alpha = 1$  at different  $kv_T/\Omega$ .

$$\mathbb{L}' b_m = \sqrt{\frac{m+1}{2}}(1 + \beta \delta_{m0})b_{m+1} + \sqrt{\frac{m}{2}}(1 + \beta \delta_{m1})b_{m-1}, \quad (41)$$

(with  $\beta = -1 + \sqrt{1 + \alpha}$ ) which also satisfies Eqn. 36. Furthermore, free energy injection, Eqn. 25, becomes simply

$$\dot{W}_S = \text{Re} \left( \sum_{m=0}^{\infty} b_m^* S'_m \right). \quad (42)$$

Thus by the same argument as before, the free energy injection in the general  $\alpha \neq 0$  case vanishes at every order in  $kv_T/\Omega$ .

## 5 Summary and Discussion

In Sec. 3 we proved a very general theorem about entropy production by a stable plasma Fourier mode. In fact, it is a simple corollary of the completeness of linear eigenmodes, proved by Case [1959], that the NEP theorem also applies to unstable wavenumbers when the system is projected into to the subspace of the stable continuum. The continuum damping in these degrees of freedom can thus be separated from the energetics of the discrete modes, which exchange energy with the background in a fundamentally reversible way. Consequently an individual Fourier mode of a plasma could avoid irreversible damping entirely via our mechanism.

In Sec. 4 we approached the problem from the perspective of a weakly collisional plasma with a Hermite-space representation. We found that the loss of damping is accompanied by a simultaneous super-algebraic decay of the Hermite spectrum. Asymptotic analysis in the fast-forcing limit demonstrate how this occurs mathematically. A practical outcome of this for physical systems with

small but finite collisions is that, at modestly small values of  $kv_T/\Omega$ , the amount of dissipation occurring becomes small, and the small amount that does occur must happen at the largest scales (lowest Hermite index  $m$ ); thus the “cascade” of energy to small scales can be effectively eliminated.

Collisionless plasma instabilities are sometimes described as “inverse Landau damping.” This is presumably due to the fact that a kind of “phase-mixing” occurs in the mechanics of an unstable plasma mode. However, the terminology is unfortunate because it neglects the fundamental reasons that Landau modes are different from these discrete modes. Discrete modes are smooth eigenmodes with velocity-space structure that is constant in time. On the other hand, Landau damping is due to fake eigenmodes that secularly develop finer-and-finer scales in velocity space. Because of this distinction, the NEP theorem reveals another weakness to conceptually lumping all plasma modes together. By simply detuning the drive frequency, the continuous components can avoid irreversible dissipation “at all orders”. What remains are discrete plasma eigenmodes that grow or decay without involving collisions.

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## A Kinetic Alfvén waves

We now consider a slight variation of Eqn. 1, also changing notation to conform with conventions. Note that Appx. A is independent of the rest of the paper, and so it should not be a significant problem if there are minor conflicts with notation that appears elsewhere. The new equation is

$$\frac{\partial f}{\partial t} + ikvf + ik \left( n - \kappa \frac{v}{v_T} j \right) \alpha g = S, \quad (43)$$

where  $g = vf_M$ , the density is defined as before (Eqn. 2), and the normalized parallel current is

$$j = \frac{1}{v_T} \int_{-\infty}^{\infty} v dv f. \quad (44)$$

This is a single-kinetic-species model that is applicable to several physical regimes, depending on how the constants  $\alpha$  and  $\kappa$  are defined. For instance, it can be applied to describe a low- $\beta$  kinetic Alfvén wave (KAW) at scales similar to the ion Larmor radius, which is also similar to the electron inertia scale; see Zocco and Schekochihin [2011] for details. In this case,  $f$  represents the perturbed distribution for the electrons, while the ions respond adiabatically. The equations could also describe electromagnetic kinetic ion dynamics near the ion inertia scale, with adiabatic electrons. In any case, we are now considering a magnetized plasma with a time-independent and spatially-uniform guide field  $\mathbf{B}_0 = B_0 \hat{\mathbf{b}}$  and a fluctuating parallel magnetic vector potential, *i.e.*

$$j = \frac{1}{\kappa \alpha} \frac{q \delta A_{\parallel}}{T} \frac{v_T}{c}. \quad (45)$$

The analysis of Eqn. 43 is more tedious than the analysis of Eqn. 1, but it is fundamentally the same. Indeed, the dimensionality of the problem is the same, *i.e.* we are solving again for  $f(k, v, t)$ , and thus an analysis in the style of Van Kampen [1955] should proceed analogously. Let us do this analysis, since it appears to have not been done elsewhere, and also because it will demonstrate the procedure for deriving a transformation like that of Morrison. Taking  $S = 0$ , we can write the eigenfunctions as

$$f^u(v) = -\alpha P \frac{g(v)}{v-u} \left( n^u - \frac{v}{v_T} \kappa j^u \right) + \lambda_u \delta(u-v). \quad (46)$$

It is convenient to take the density moment of Eqn. 43 to obtain a relationship between  $n^u$  and  $j^u$ :

$$j^u = -\frac{\gamma}{\kappa} \frac{u}{v_T} n^u, \quad (47)$$

where  $\gamma = 2\kappa/(\alpha\kappa - 2)$ . Then we take  $n^u = 1$  as a choice of normalization. This leads to

$$f^u(v) = -\alpha P \frac{g(v)}{v-u} \left( 1 + \frac{uv}{v_T^2} \gamma \right) + \lambda_u \delta(u-v), \quad (48)$$

and

$$\lambda_u = 1 - \alpha \pi \left( 1 + \frac{u^2}{v_T^2} \gamma \right) g_*, \quad (49)$$

where we have used identity 61 to obtain Eqn. 49. Now we demonstrate that these eigenfunctions are complete, *i.e.* that any function  $f(v)$  can be written as

$$f(v) = \int_{-\infty}^{\infty} du C(u) f^u(v). \quad (50)$$

The task is to invert this expression for  $C(u)$ . This will simultaneously (1) prove completeness of the eigenfunctions and (2) give us the transformation that we seek along with its inverse. Substituting in Eqn. 48 we find

$$-\alpha \pi g \left[ C_* + \frac{\gamma v}{v_T^2} C'_* \right] + \lambda_v C = f \quad (51)$$

where we have notated  $C'(v) = vC(v)$ . Using again identity 61 and substituting in Eqn. 49 we obtain

$$\alpha g \frac{v}{v_T} \gamma \bar{C} - \alpha \pi \left( 1 + \frac{v^2}{v_T^2} \gamma \right) [gC_* + g_* C] + C = f, \quad (52)$$

where we have introduced  $\bar{C} = v_T^{-1} \int dv C(v)$ . Then we expand  $f = f_+ + f_-$ ,  $C = C_+ + C_-$ , use the identity 60, and  $v^2 g_*(v) = (v^2 g)_* + v_T^2/(2\pi)$ . This results in only terms which are positive- and negative-frequency functions, and so can be equated separately. This yields the two equations

$$\pm 2\pi i \alpha g'_\pm C_\pm + \left( 1 - \frac{\alpha \gamma}{2} \right) C_\pm + \bar{C} H_\pm = f_\pm, \quad (53)$$

where we have defined  $g' = (1 + \gamma v^2/v_T^2)g$  and  $H = \alpha\gamma(1 + \gamma v^2/v_T^2)g/v_T$ . Solving Eqn. 53 for  $C_\pm$  we have

$$C_\pm = \frac{f_\pm - \bar{C}H_\pm}{D_\pm}, \quad (54)$$

where we define

$$D_\pm = 1 - \alpha\gamma/2 \pm 2\pi i\alpha g'_\pm. \quad (55)$$

What is left is to solve for  $\bar{C}$ , which can be done by integrating Eqn. 54 over  $v$ . To express this cleanly, let us introduce the auxiliary transformation<sup>2</sup>

$$\check{f} = \frac{f_+}{D_+} + \frac{f_-}{D_-}. \quad (56)$$

Then, using  $\int dv C(v) = 2 \int dv C_\pm(v)$ , Eqn. 54 leads to

$$\bar{C} = \frac{\frac{2}{v_T} \int dv \frac{f_\pm}{D_\pm}}{1 + \frac{2}{v_T} \int dv \frac{H_\pm}{D_\pm}} = \frac{\check{f}}{1 + \check{H}}. \quad (57)$$

Completeness of the eigenfunctions  $f^u$  is thus proved (by Eqns. 54 and 57) if  $C_+$  is a positive frequency function, which is ensured if the denominator in Eqn. 54 (*i.e.*  $D_+$ ) has no zeros in the upper half plane. This is satisfied if and only if there are no unstable solutions to Eqn. 43. (This can be shown by Nyquist diagram, *i.e.* by demonstrating both the dispersion relation and the denominator  $D_+$  have the same image under mapping to the upper half plane.) Finally, as promised, Eqn. 54 immediately provides the transformation via  $\check{f}(u) \equiv C(u) = C_+(u) + C_-(u)$ , *i.e.*

$$\check{f} = \check{f} - \check{H} \frac{\check{f}}{1 + \check{H}}, \quad (58)$$

which can be inverted using Eqn. 50 by substituting  $C(u) = \check{f}(u)$ . This completes the derivation of the transformation for our system. What is immediately implied is that Eqn. 43 is equivalent to Eqn. 3, where the transformation is the new one, Eqn. 58. Thus, with the appropriate substitutions, Eqns. 10-15 will also apply, along with their physical implications.

## B Identities

A useful identity relating the Hilbert transform to the positive/negative frequency parts is

$$h_\pm = \frac{1}{2}(h \pm i h_*), \quad (59)$$

implying also

$$h_* = -i(h_+ - h_-). \quad (60)$$

<sup>2</sup> The invertibility of the auxiliary transformation follows from the invertibility of Eqn. 58. Furthermore, invertibility implies that it satisfies an equation analogous to Eqn. 62, which is used between the first and second equality in Eqn. 57.

Integration of Eqn. 6 by parts yields the identity

$$(vh)_* = v h_* - \frac{1}{\pi} \int_{-\infty}^{\infty} h(v') dv', \quad (61)$$

We will also need the following identity, which follows from the invertibility of the Morrison transform

$$(\check{f})_\pm = \frac{f_\pm}{D_\pm}. \quad (62)$$

## C Derivation of rate of free energy input

To compute the rate of free energy input at steady state, we start with the energy budget equation. First we multiply Eqn. 1 by  $f^*v/G = f^*v_T^2/(2f_M)$ , integrate over velocity and take the real part. Next we multiply Eqn. 1 by  $\alpha n^*/2$ , where  $\alpha = 2c_s^2/v_T^2$ , integrate over velocity, and take the real part. The sum of the two equations yields

$$\frac{dW}{dt} = \text{Re} \left[ e^{-i\Omega t} \left( \int \frac{f^* S(v)}{f_M} dv + \alpha n^* \int dv S(v) \right) \right]. \quad (63)$$

Now taking  $S(v) = S_0 \varphi(v) f_M$ , and using the long-time solution of  $f$ , Eqn. 10, we obtain Eqn. 17 with

$$H = \tilde{s} \{ \alpha + \varphi^* + \alpha \pi ((\varphi^* f')_* - \varphi^* f'_*) \}. \quad (64)$$

where  $s = S(v)/S_0 = \varphi f_M$ . We will now manipulate Eqn. 64 in a series of steps, until we ultimately obtain Eqn. 18. It is tedious but straightforward. We will need a few identities along the way. First, for any  $h(v)$  and  $\varphi = \sum_{m=0}^N \varphi_m v^m$  we have

$$(\varphi h)_* = \varphi h_* - \frac{1}{\pi} \Phi[\varphi, v h], \quad (65)$$

where we define the polynomial

$$\Phi[\varphi, h](u) = \sum_{m=0}^{N-1} T_{m+1}[\varphi] \frac{1}{u^{m+1}} \int dv h(v) v^m, \quad (66)$$

and the truncated polynomial

$$T_m[\varphi](u) = \sum_{n=m}^N \varphi_n u^n. \quad (67)$$

Now using Eqn. 65 we may rewrite Eqn. 64 as

$$H = \tilde{s} \{ \varphi^* + \alpha (1 - \Phi[\varphi^*, f'_M]) \}, \quad (68)$$

where  $f'_M(v) = v f_M$ . It can be shown that

$$1 - \Phi[\varphi^*, f'_M] = \varphi^* - u \Phi^*, \quad (69)$$

where we have introduced the abbreviation  $\Phi[\varphi, f_M] = \Phi$ . Eqn. 68 now becomes

$$H = \tilde{s} \{ (1 + \alpha) \varphi^* - \alpha u \Phi^* \}. \quad (70)$$



Now from Eqn. 65 (and Eqn. 59) we find

$$\tilde{s} = \widetilde{f_M \varphi} = \tilde{f}_M \varphi - \frac{i}{2\pi} Q P, \quad (71)$$

where

$$Q = \frac{1}{D_+} - \frac{1}{D_-}, \quad (72)$$

$$= -2\pi i \alpha \tilde{f}'_M, \quad (73)$$

$$= -2\pi i \alpha u \tilde{f}_M / (1 + \alpha). \quad (74)$$

Using Eqns. 71 and 74 then leads to the form

$$H = \psi(u) \tilde{f}_M, \quad (75)$$

where

$$\psi = (1 + \alpha)|\varphi|^2 + u^2 \frac{\alpha^2}{1 + \alpha} |\Phi|^2 - \alpha u (\varphi \Phi^* + \varphi^* \Phi), \quad (76)$$

is a purely real polynomial. Now, using Eqn. 65, we can write

$$H = \widetilde{\psi f_M} + \frac{i}{2\pi} Q \Phi[\psi, f_M]. \quad (77)$$

Finally, we take the positive frequency part of this equation, apply Eqn. 62 and Eqn. 65 once more, and find

$$H_+ = \frac{(\psi f_M)_+}{D_+} + \frac{i}{2\pi} Q_+ \Phi[\psi, f_M] - \left( \frac{i}{2\pi} \right)^2 \Phi[\Phi[\psi, f_M], Q]. \quad (78)$$

Because  $Q$  is purely imaginary, the final term can be neglected in computing  $\text{Re}[H_+]$ . Using Eqn. 73 we then obtain Eqn. 18.